

The Free Energy of High Temperature QED to Order e^5 From Effective Field Theory

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Abstract

Massless quantum electrodynamics is studied at high temperature and zero chemical potential. We compute the Debye screening mass to order e^4 and the free energy to order e^5 by an effective field theory approach, recently developed by Braaten and Nieto. Our results are in agreement with calculations done in resummed perturbation theory. This method makes it possible to separate contributions to the free energy from different momentum scales (order T and eT) and provides an economical alternative to computations in the full theory which involves the dressing of internal propagators.

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1 Introduction

Effective field theories both at zero and finite temperature have become of increasing interest in recent years [1-3]. Modern developments in renormalization theory have given meaning to non-renormalizable quantum field theories (“effective theories”) and it is no longer obvious that renormalizability is an essential property of a useful field theory [4]. The distinction between effective theories and “fundamental” (renormalizable) field theories becomes rather vague. Indeed, the modern view is to consider every field theory as effective, valid and with predictive power at a certain scale.

There has been a tremendous progress in perturbative calculations of the thermal properties for quantum field theories at high temperatures since the pioneering work of Dolan and Jackiw some twenty years ago [5]. Today, one can essentially distinguish between two methods for computing high temperature properties of a quantum field theory. The first method is based on resummed perturbation theory. Resummation is a reorganization of the perturbation series in order to be

actually perturbative. Resummation gives rise to effective thermal masses (and in non-Abelian theories effective vertices, too) that provide an infrared cut off in the loops. The dressed propagators are thus well behaved and the use of these in loops yields the contributions to the free energy that are non-analytic in g^2 , where g is the coupling constant. The use of resummed perturbation theory amounts to summing an infinite set of higher order diagrams. This resummation scheme has largely been developed by Braaten and Pisarski [6] and has been applied to QCD by a number of authors in connection with the gauge dependence of the gluon damping rate and the screening mass (Refs. [7,8] and references therein). The method has also been used to calculate the free energy in QED [9,10] and in the standard model [11].

The second method is based on dimensional reduction [12]. The crucial observation is that the only contribution to a correlator that is not exponentially damped at scales larger than T comes from the static bosonic modes. One strategy is therefore to integrate out the non-static bosonic modes as well as the fermionic modes to construct an effective three-dimensional field theory of the zero modes, in which the masses and couplings are temperature dependent. This approach has been used in the study of high temperature QCD and QED [12] as well as in investigations of phase transitions in spontaneously broken gauge theories [1,13] (in particular the electro-weak phase transition).

Instead of explicitly integrating over the heavy modes (which produces non-local terms beyond one-loop [14]), one simply writes down the most general three-dimensional Lagrangian, that respects the symmetries of the system. One then computes static correlators in the two theories and require that they match. This matching requirement can actually be taken as the definition of the effective field theory and provides the relationship between the coupling constants in the effective theory and the underlying theory. This method has previously been applied to massless ϕ^4 theory [2] and QCD [15]. In the latter case it does not only provide a convenient way of calculating the free energy in the high temperature limit, it does also solve the infrared catastrophe of Non-Abelian gauge theories. It is a well known fact that the free energy of Non-Abelian gauge theories may be calculated to fifth order in the coupling using resummed perturbation theory [16]. However, the method breaks down at order g^6 and this has been interpreted as a sign that the transverse gluons have a mass of order $g^2 T$, which cannot be calculated in perturbation theory [11]. This magnetic screening mass should then provide the necessary infrared cutoff.

In the present paper we would like to apply the method to QED and confirm some recent results obtained by calculations in the full theory [9,10]. Our work will serve as a nice demonstration of the efficiency of the effective field theory approach by allowing one to work with a single scale at a time.

The outline of this work is as follows: In section two we compute the screening mass to fourth order and the free energy to fifth order in the coupling constant in high temperature QED. In section three we summarize. Throughout the work we use the imaginary time formalism and the theory is regularized by working in $4 - 2\epsilon$ dimensions together with the \overline{MS} regularization scheme. We use the following shorthand notation for the sum-integrals that appear below:

$$\int \{dK\} f(K) \equiv \mu^{2\epsilon} T \sum_{k_0=2\pi(n+1/2)T} \int \frac{d^{3-2\epsilon} k}{(2\pi)^{3-2\epsilon}} f(K), \quad (1)$$

where $K^2 = k_0^2 + k^2$. In the Feynman diagrams a solid lines denotes a fermion, a wavy line a photon and a dashed line a ghost.

2 QED at High Temperature

The partition function can be written as a path integral

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[- \int_0^\beta d\tau \int d^3x \mathcal{L} \right], \quad (2)$$

where the Lagrangian of QED reads

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} \gamma^\mu (\partial_\mu - ie A_\mu) \psi + \mathcal{L}_{\text{gf}}. \quad (3)$$

In this paper we choose to work in Feynman gauge where $\mathcal{L}_{\text{gf}} = \frac{1}{2}(\partial_\mu A_\mu)^2$. In the effective theory the partition function reads

$$\mathcal{Z} = e^{-f(\Lambda)V} \int \mathcal{D}A_i \mathcal{D}A_0 \exp \left[- \int d^3x \mathcal{L}_{\text{eff}} \right]. \quad (4)$$

Here \mathcal{L}_{eff} is the effective three-dimensional Lagrangian to which we shall return shortly. The prefactor $f(\Lambda)$ can be interpreted as the coefficient of the unit operator in the effective field theory. It depends on the ultraviolet cutoff Λ (which is introduced to regularize the effective three-dimensional theory) in order to cancel the Λ dependence in the path integral in Eq. (4) [2]. The effective three-dimensional theory consists of a gauge field and a real massive self interacting scalar field in the adjoint representation of the gauge group:

$$\mathcal{L} = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (\partial_i \rho)^2 + \frac{1}{2} m^2(\Lambda) \rho^2 + \lambda(\Lambda) \rho^4 + \mathcal{L}_{\text{gf}} + \delta \mathcal{L}. \quad (5)$$

At this point a few comments are in order. Firstly, we remark that the scalar field ρ is identified with the time component A_0 of the gauge field in the full theory, up to

normalizations. This implies that the field A_0 does not interact with the gauge field in the effective theory, since QED is an Abelian gauge theory. The fact that A_0 may develop a thermal mass is simply a consequence of the lack of Lorentz invariance at non-zero temperature. Secondly, it can easily be demonstrated from the matching of four-point functions that $\lambda(\Lambda)$ is of order e^4 and that the term $\lambda(\Lambda)\rho^4$ does not contribute to the Debye mass or the free energy to the order we are calculating (This interaction contributes at order e^5 to the screening mass and at order e^6 to the free energy).

Furthermore, $\delta\mathcal{L}$ represents all local terms which respect the symmetries of the theory such as gauge invariance. This includes renormalizable terms, such as $g(\Lambda)\rho^6$, as well as non-renormalizable ones.

The mass parameter. In order to obtain the free energy to order e^5 , we need to compute the short distance coefficient $m^2(\Lambda)$ to order e^4 . The e^3 and e^5 contributions to $\ln \mathcal{Z}$ is namely given by a one-loop calculation in the effective theory (see below). The simplest way to determine $m^2(\Lambda)$ in the effective theory is by demanding that the screening mass in the two theories match. The screening mass of the particles is defined as the location of the pole of the propagator for spacelike momentum [2]:

$$p^2 + \Pi_{00}(0, \mathbf{p}) = 0, \quad p^2 = -m_s^2. \quad (6)$$

This definition is gauge fixing independent, which one can prove on an algebraic level [17]. This is in contrast with the one found in the book by Kapusta [18] and which commonly used in the literature¹:

$$m_s^2 = \Pi_{00}(p_0 = 0, \mathbf{p} \rightarrow 0). \quad (7)$$

Normally, as in the present case, the two definitions are equivalent to leading order in the coupling constant. Beyond leading order they do not coincide. The requirement above implies that

$$p^2 + m^2(\Lambda) + \Pi_{\text{eff}}(p, \Lambda) = 0, \quad p^2 = -m_s^2, \quad (8)$$

where $\Pi_{\text{eff}}(p, \Lambda)$ is the self-energy of ρ in the effective theory. We shall do the matching in strict perturbation theory. Generally, this means that the expression for the screening mass differs from the one obtained in resummed perturbation theory, which correctly incorporates the effects of electrostatic screening. The solution m_s^2 is of order e^2 , which implies that one can expand $\Pi(p^2) \equiv \Pi_{00}(0, p)$ in a Taylor series around $p^2 = 0$. To determine the screening mass (in strict perturbation theory) to

¹In quantum electrodynamics the latter *is* gauge fixing independent, but it is *not* renormalization group invariant. In non-Abelian theories it is also gauge dependent. This is to be expected since it is an off-shell quantity [7].

order e^4 , we must calculate $\Pi(p^2)$ to one loop order and $\Pi'(0)$ to two loop order, and the screening mass is then given by

$$m_s^2 \approx \Pi(0) [1 - \Pi'(0)]. \quad (9)$$

The symbol \approx indicates that Eq. (9) only holds in strict perturbation theory. Now, strict perturbation theory in the effective theory means that the mass term should be treated as a perturbation. The corresponding contribution to Π_{eff} is shown in Fig. 1, where the blob indicates the mass insertion. The matching relation simply becomes $m^2(\Lambda) \approx m_s^2$.

The self-energy to one-loop order in the full theory reads

$$\begin{aligned} \Pi(p^2) &= e^2 \int \{dK\} Tr(\gamma_0 \frac{1}{K} \gamma_0 \frac{1}{K + P}) \\ &= -e^2 \int \{dK\} \frac{8}{K^2} + e^2 \frac{4}{3} p^2 \int \{dK\} \frac{1}{K^4} + \mathcal{O}(p^4). \end{aligned} \quad (10)$$

The corresponding Feynman diagram is shown in Fig. 2. The sum-integrals in Eq. (10) are standard and can be found in e.g. Ref. [19]. Carrying out the renormalization of the wave function in the usual way, one finds:

$$\Pi(p^2) = \frac{e^2 T^2}{3} + \frac{p^2}{12\pi^2} (2\gamma_E - 1 + 2 \ln \frac{\Lambda}{4\pi T}). \quad (11)$$

Here Λ is the scale introduced by dimensional regularization. The two-loop expression for the self energy at zero external momentum can be found either by a direct computation of the two-loop graphs or by applying the formula (see Ref. [18])

$$\Pi(0) = -e^2 \frac{\partial^2 P}{\partial \mu^2}, \quad (12)$$

where P is the pressure and μ is the chemical potential. This requires the calculation of the free energy to two-loop order including the chemical potential. This can be carried out by contour integration [18]. Using the result $-T^2 e^4 / 8\pi^2$ for the e^4 contribution, we find

$$\Pi(p^2) = \frac{e^2 T^2}{3} + \frac{e^2 p^2}{12\pi^2} (2\gamma_E - 1 + 2 \ln \frac{\Lambda}{4\pi T}) - \frac{e^4}{8\pi^2}. \quad (13)$$

The mass parameter to order e^4 then becomes:

$$m^2(\Lambda) = T^2 \left[\frac{e^2}{3} - \frac{e^4}{36\pi^2} (2\gamma_E - 1 + 2 \ln \frac{\Lambda}{4\pi T}) - \frac{T^2 e^4}{8\pi^2} \right]. \quad (14)$$

By using the renormalization group equation for the coupling constant,

$$\mu \frac{de^2}{d\mu} = \frac{e^4}{6\pi^2} + \mathcal{O}(e^6), \quad (15)$$

one can easily demonstrate that Eq. (14) is independent of Λ . Thus, up to corrections of order e^6 , we can replace Λ by an arbitrary renormalization scale μ .

The coupling constant. We would also like to make a few remarks about the gauge coupling e_3 in the effective theory. Using the relation between the gauge fields in the two theories

$$A_i^3 = \sqrt{T} A_i, \quad (16)$$

one finds that $e_3^2 = T e^2$ to leading order in e^2 . There is no dependence of Λ at this order. Beyond leading order one must compute and match four point correlation functions [2].

The coefficient of the unit operator. We shall now compute $f(\Lambda)$ to order e^4 in strict perturbation theory. A strict perturbative expansion in the coupling constant is normally plagued by severe infrared divergences, which are caused by long-range forces. Physically, these forces are screened at the length scale $1/(eT)$ and can be taken into account only by reorganizing perturbation theory which amounts to summing an infinite set of loops. Nevertheless, the strict perturbation theory can be used to compute the short distance coefficient $f(\Lambda)$ and we shall do so by matching calculations of $\ln \mathcal{Z}$ in the full theory and in the effective theory. From eqs. (2) and (4), we see that the matching condition reads

$$\ln \mathcal{Z} = -f(\Lambda)V + \ln \mathcal{Z}_{\text{eff}}. \quad (17)$$

The calculation of $\ln \mathcal{Z}$ in the full theory involves one-loop, two-loop and three-loop diagrams. The graphs are displayed in Figs. 3–5. The diagrams can be computed by the methods developed in Refs. [9,19] and details may be found there. We find:

$$\begin{aligned} \frac{T \ln \mathcal{Z}}{V} \approx & \frac{\pi^2 T^4}{9} \left[\frac{11}{20} - \frac{5e^2}{32\pi^2} + \frac{e^4}{256\pi^4} \left(-\frac{20}{3} \ln \left(\frac{\Lambda}{4\pi T} \right) + \frac{8}{3} \frac{\zeta'(-3)}{\zeta(-3)} - \frac{16}{3} \frac{\zeta'(-1)}{\zeta(-1)} \right. \right. \\ & \left. \left. - 4\gamma_E - \frac{319}{12} + \frac{208}{5} \ln 2 \right) \right]. \end{aligned} \quad (18)$$

Here \approx again indicates that Eq. (18) only holds in strict perturbation theory, and Λ is the momentum scale which is introduced by dimensional regularization. Λ may again be traded for an arbitrary μ by RG-arguments.

We now turn to the effective theory. The mass parameter is viewed as a perturbation in the effective theory. $\ln \mathcal{Z}_{\text{eff}}$ is then given by one-loop contributions from

the gauge field and scalar field (Figs. 3–5), plus an additional one-loop diagram with a mass insertion (which is indicated by a blob in Fig. 6). The computation is rather simple since one-loop contributions involving massless fields vanish identically in dimensional regularization (see Eq. (21) below) and so does $\ln \mathcal{Z}_{\text{eff}}$. The matching condition therefore turns out to be

$$\frac{T \ln \mathcal{Z}}{V} \approx -f(\Lambda)T, \quad (19)$$

and $f(\Lambda)$ is given by the right hand side of Eq. (18). The function $F = f(\Lambda)T$ can be viewed as the contribution to the free energy from the short distance scale $1/T$. With the comments after Eq. (18) in mind, it is clear that $f(\Lambda)$ has no dependence of Λ at the order we are calculating.

Now that we have determined the short distance coefficients $m^2(\Lambda)$ and $f(\Lambda)$ in the effective theory, we can calculate the screening mass as well as the free energy in QED to order e^4 and e^5 , respectively.

The screening mass. In order to calculate the *physical* screening mass, one must include the mass parameter $m^2(\Lambda)$ in the free part of the Lagrangian. Furthermore, since the term $\lambda(\Lambda)$ goes like e^4 , the ϕ^4 term does not contribute to the screening mass to order e^4 , as we mentioned above. This implies that the physical screening mass² is equal to the short distance coefficient $m^2(\Lambda)$ to order e^4 :

$$m_s^2 = T^2 \left[\frac{e^2}{3} - \frac{e^4}{36\pi^2} (2\gamma_E - 1 + 2 \ln \frac{\bar{\mu}}{4\pi T}) - \frac{T^2 e^4}{8\pi^2} \right]. \quad (20)$$

It is easily checked that the result is RG-invariant as required. Furthermore, our result agrees with the calculation of Blaizot *et al* to order e^4 [20]. Note also that there is no e^3 term in the expression for the screening mass in contrast with both ϕ^4 theory and SQED. The reason is that there are no bosonic propagators in the one-loop self-energy graph in QED and fermions need no resummation, since their Matsubara frequencies are never zero.

The free energy. The calculation of the free energy in the effective theory is now straightforward, it is simply a one-loop computation. However, we must now include the physical effect of screening, which amounts to consider the mass parameter as a term in the free part of the Lagrangian. Using dimensional regularization (see e.g. Ref. [21]) the one-loop integrals are perfectly finite after regularization and

²It is also equal to the screening mass obtained in the strict perturbation theory to order e^4 . This is due to the fact that there is no e^3 term, as explained above. At order e^5 and higher they differ, caused by the dressing of bosonic propagators in the two-loop diagrams.

independent of the renormalization scale:

$$\int \frac{d^3k}{(2\pi)^3} \ln(k^2 + M^2) = -\frac{1}{6\pi} M^3. \quad (21)$$

The contribution from the gauge field vanishes. Using the expression for the mass of the scalar field and expanding it in powers of e yields the following contribution to the free energy:

$$\ln \mathcal{Z}_{\text{eff}} = T^4 \left[\frac{e^3}{36\sqrt{3}\pi} - \frac{e^5}{576\sqrt{3}\pi^3} \left(4 \ln \frac{\bar{\mu}}{4\pi T} + 4\gamma_E + 8 \ln 2 + 9 \right) \right]. \quad (22)$$

This term takes into account the effects from long distance scales of order $1/(eT)$. Note that it is non-analytic in e^2 . Furthermore, in resummed perturbation theory the e^3 and the e^5 terms would arise from the dressing of the photon propagator in the two and three-loop diagrams, respectively. Putting the results from Eq.s (18) and (22) together, one obtains

$$\begin{aligned} \ln \mathcal{Z} = & T^4 \left[\frac{11\pi^2}{180} - \frac{5e^2}{248} + \frac{e^3}{36\sqrt{3}\pi} + \frac{e^4}{2304\pi^2} \left(-\frac{20}{3} \ln \left(\frac{\bar{\mu}}{4\pi T} \right) + \frac{8}{3} \frac{\zeta'(-3)}{\zeta(-3)} - \frac{16}{3} \frac{\zeta'(-1)}{\zeta(-1)} \right. \right. \\ & \left. \left. - 4\gamma_E - \frac{319}{12} + \frac{208}{5} \ln 2 \right) - \frac{e^5}{576\sqrt{3}\pi^3} \left(4 \ln \frac{\bar{\mu}}{4\pi T} + 4\gamma_E + 8 \ln 2 + 9 \right) \right]. \end{aligned}$$

This is the main result of the present paper and it is in agreement with the computation of Zhai and Kastening [9], and Parwani [10] who use resummed perturbation theory. Eq. (23) is correct as well as renormalization group invariant up to corrections of order $e^6 \ln e$. The latter property may easily be checked by using the one-loop β -function in QED.

Finally, we would like to comment upon a computational difference between QED and QCD. In QCD the computation of the free energy involves the construction of *two* effective field theories, which reflects the fact that there are contributions from three different momentum scales (T , gT and g^2T , where g is the gauge coupling) [15]. The first effective field theory, called electrostatic QCD (EQCD), consists of the magnetostatic A_i^a field and the electrostatic field A_0^a . The unit operator f_{QCD} as well as the mass parameter in EQCD is then determined by the matching procedure and f_{QCD} gives the contribution to the free energy from the short distance scale $1/T$. The second effective field theory is called magnetostatic QCD (MQCD) and consists simply of the self interacting magnetostatic gauge field A_i^a . Again the unit operator of this effective theory, f_{EQCD} , can be determined and yields the contribution to the free energy from the distance scale $1/(gT)$. Now, the perturbative expansion in MQCD is plagued with infrared divergences, implying that the functional integral can only be calculated non-perturbatively, e.g. by putting MQCD on a lattice.

Using lattice simulations the path integral may be computed, so that one obtains the contribution to the free energy from the scale $1/(g^2 T)$. One can, of course, also construct a second effective field theory in QED, but it is completely unnecessary. The restriction to the magnetostatic zero modes simply yields a free photon field theory in three dimensions. This in turn implies that there are no contribution from momentum scales $e^2 T$ in QED. The same remark applies to scalar electrodynamics.

3 Summary

In this work we have computed the free energy at high temperature in massless quantum electrodynamics to order e^5 using an effective field theory approach. This approach makes it possible to separate contributions which come from different length scales ($1/T$ and $1/(eT)$). Furthermore, this method involves considerably less effort in obtaining the free energy to a given order in perturbation theory than computations in the full four dimensional theory.

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FIGURE CAPTIONS:

Figure 1: Self-energy correction in the effective theory.

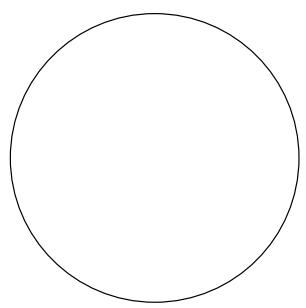
Figure 2: One-loop self-energy correction.

Figure 3: One-loop diagrams.

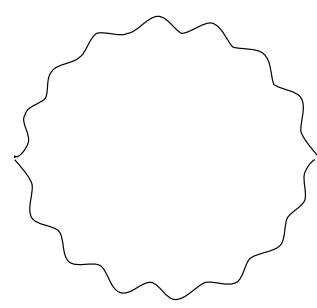
Figure 4: Two-loop diagram.

Figure 5: Three-loop diagrams.

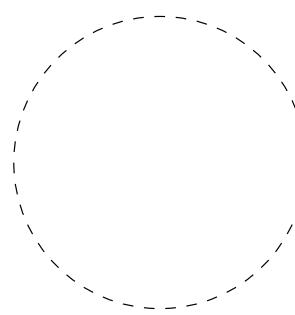
Figure 6: One-loop diagram with mass insertion.



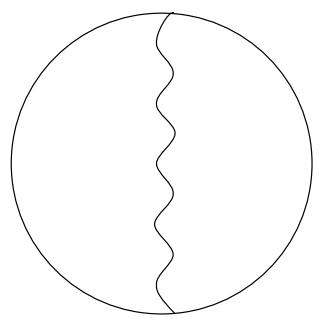
(a)



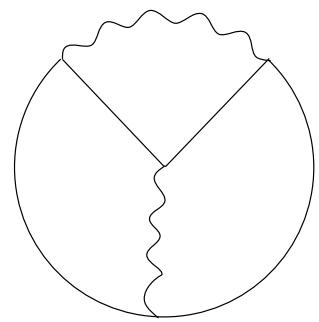
(b)



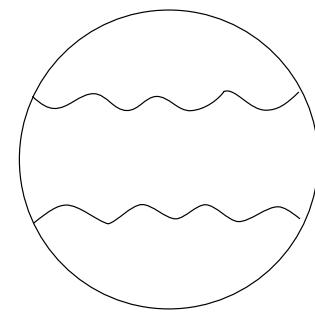
(c)







(a)



(b)

